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Quasi-Periodic Solution of the Kadomtsev-Petviashvili Equation

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Abstract

The Kadomtsev-Petviashvili (KP) equation is separated into systems of compatible ordinary differential equations with the help of two (1+1)-dimensional soliton equations. Quasi-periodic solution of the KP equation is finally obtained in terms of Riemann theta functions. During that course, the generating function approach is used to prove the involutivity and the functional independence of the conserved integrals, and the Abel-Jacobi coordinates are introduced to linearize the associated flows.

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1. Introduction

Soliton equations are important models describing nonlinear phenomena that occur in nature, and have many applications in various fields of physical sciences such as nonlinear waves, nonlinear optics, plasma physics, and magnetic fluids. The study of explicit solutions for various soliton equations has been very important in modern mathematics with ramifications to several areas of mathematics, physics and other sciences. Nowadays, many systematic methods have been developed to derive the explicit solutions of soliton equations, for instance, the inverse scattering transformation [1,2], the algebro geometric method [3-5], the nonlinearization approaches of Lax pairs [6-11], the Hirota bilinear method [12] and others [13-15]. The nonlinearization approach of eigenvalue problems or Lax pairs is a powerful tool in the study of (1+1)- and (2+1)-dimensional soliton equations. This technique solves (1+1)-dimensional

soliton equations by decomposing them into the compatible ordinary differential equations, which are the finite dimensional completely integrable systems in Liouville sense. The (2+1)-dimensional soliton equations could be decomposed into the (1+1)-dimensional soliton equations in a similar procedure [16-19], and further into the compatible ordinary differential equations.

In the present paper, we use the method given in [7-10] to construct the quasi-periodic solution of the KP equation and associated (1+1)-dimensional soliton equations. The outline of this paper is as follows. In section 2, the KP equation is decomposed into two compatible (1+1)-dimensional soliton equations. In section 3, the Bargmann constraint between the potentials and eigenfunctions is given to derive a finite-dimensional Hamiltonian system. In section 4, the (1+1)-dimensional soliton equations and the KP equation are separated into the compatible Hamiltonian systems of ordinary differential equations. In section 5, elliptic variables and quasi-Abel-Jacobi coordinates are introduced to prove the integrability of the finite-dimensional Hamiltonian system. In section 6, the quasi-periodic solutions are obtained by means of Riemann theta functions resorting to the Abel-Jacobi coordinates.

2. Lenard Gradients and the Soliton Hierarchy

Let us begin with the problem as follows:

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\lambda + u & 2v \\ 2(v^2 + \frac{\gamma_1}{v}) & \lambda - u \end{pmatrix}, \quad \phi = \begin{pmatrix} p \\ q \end{pmatrix}, \quad (1)$$

where λ is a constant spectral parameter, and u, v are two potentials. We have

$$V_x - [U, V] = \sigma_\lambda[(K - \lambda J)g_\lambda], \quad (2)$$

Where

$$V = \begin{pmatrix} C & B \\ A & -C \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \partial + 2u & 0 & -4(v^2 + \frac{\gamma_1}{v}) \\ 0 & \partial - 2u & 4v \\ -2v & 2(v^2 + \frac{\gamma_1}{v}) & \partial \end{pmatrix},$$

$$\partial = \partial / \partial x, \partial \partial^{-1} = \partial^{-1} \partial = I, g_\lambda = (A, B, C)^T, \quad \sigma_\lambda : (A, B, C)^T \mapsto \begin{pmatrix} C & B \\ A & -C \end{pmatrix}.$$

Consider $g_\lambda = \sum_{n \geq 0} g_{n-1} \lambda^{-n}$, where $g_n = (g_n^{(1)}, g_n^{(2)}, g_n^{(3)})^T$ and

$$\begin{cases} g_n^{(1)} = \frac{1}{2} \partial g_{n-1}^{(1)} + u g_{n-1}^{(1)} - 2(v^2 + \frac{\gamma_1}{v}) g_{n-1}^{(3)}, \\ g_n^{(2)} = -\frac{1}{2} \partial g_{n-1}^{(2)} + u g_{n-1}^{(2)} - 2v g_{n-1}^{(3)}, \\ g_n^{(3)} = \frac{1}{2} \sum_{\substack{j+k=n-1 \\ j,k \geq 0}} g_j^{(1)} g_k^{(2)} + g_j^{(3)} g_k^{(3)}, \quad g_0^{(3)} = 0. \end{cases} \quad (3)$$

Then by (2), $V = \sigma_\lambda(g_\lambda)$ solves the Lax equation

$$V_x - [U, V] = 0, \quad (4)$$

if we define the Lenard recursive relation $Kg_{n-1} = Jg_n, n \geq 0$. Together with (3) and the given initial values $g_{-1} = (0, 0, -1)^T$, we can recursively establish $g_n, n \geq 0$.

Assume that the time dependence of ϕ for the problem (1) obeys the differential equation

$$\phi_{t_n} = V^{(n)} \phi, \quad V^{(n)} = \sigma_\lambda(G_n), \quad (5)$$

where

$$\begin{aligned} G_n &= \sum_{j=0}^n g_{j-1} \lambda^{n-j} + (0, 0, \rho_n)^T, \\ \rho_n &= \frac{1}{6v^2} [g_n^{(1)} + (2v - \frac{\gamma_1}{v^2}) g_n^{(2)}]. \end{aligned}$$

Then the compatibility condition of (1) and (5) is the zero-curvature equation, $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$, which is equivalent to

$$\begin{cases} u_{t_n} = \frac{1}{6} \partial [\frac{1}{v^2} g_n^{(1)} + (\frac{2}{v} - \frac{\gamma_1}{v^4}) g_n^{(2)}], \\ v_{t_n} = \frac{1}{3v} g_n^{(1)} - \frac{1}{3} (1 + \frac{\gamma_1}{v^3}) g_n^{(2)}. \end{cases}$$

So we can define the vector fields $\{X_n\}$ by

$$X_n = \begin{pmatrix} \frac{1}{6} \partial \frac{1}{v^2} & \frac{1}{6} \partial (\frac{2}{v} - \frac{\gamma_1}{v^4}) \\ \frac{1}{3v} & -\frac{1}{3} (1 + \frac{\gamma_1}{v^3}) \end{pmatrix} \begin{pmatrix} g_n^{(1)} \\ g_n^{(2)} \end{pmatrix} = QP g_n, \quad (6)$$

where P is the projective map $(a_1, a_2, a_3)^T \mapsto (a_1, a_2)^T$, and

$$Q = \begin{pmatrix} \frac{1}{6} \partial \frac{1}{v^2} & \frac{1}{6} \partial \left(\frac{2}{v} - \frac{\gamma_1}{v^4} \right) \\ \frac{1}{3v} & -\frac{1}{3} \left(1 + \frac{\gamma_1}{v^3} \right) \end{pmatrix}.$$

If we adopt $t_1 = x, t_2 = y, t_3 = t$, according to (6), we have

$$\begin{aligned} u_y &= \left(\frac{1}{3v} - \frac{\gamma_1}{6v^4} \right) v_{xxx} + \frac{\gamma_1}{v^5} v_{xx} v_x - \left(\frac{1}{3v^3} + \frac{5\gamma_1}{6v^6} \right) v_x^3 \\ &\quad - 6v^2 v_x - \frac{\gamma_1}{v^4} u_x v_x - \left(\frac{1}{6} - \frac{\gamma_1}{3v^3} \right) u_{xx} + 2uu_x, \\ v_y &= \frac{1}{6} \left(1 - \frac{2\gamma_1}{v^3} \right) v_{xx} + \frac{1}{3v} \left(1 + \frac{\gamma_1}{v^3} \right) v_x^2 \\ &\quad + 2uv_x + \frac{2}{3} \left(v + \frac{\gamma_1}{v^2} \right) u_x, \end{aligned} \quad (7)$$

$$\begin{aligned} u_t &= 6u \left(2v - \frac{\gamma_1}{v^2} \right) v_{xxx} - \frac{18\gamma_1}{v^4} v_x^2 v_{xx} + 3 \left(1 + \frac{\gamma_1}{v^3} \right) v_{xx}^2 \\ &\quad + (24u + 3) \left(1 + \frac{\gamma_1}{v^3} \right) v_x v_{xx} + \frac{12\gamma_1}{v^5} v_x^4 - \frac{18\gamma_1 u}{v^4} v_x^3 \\ &\quad + 12 \left[u_x \left(2v - \frac{\gamma_1}{v^2} \right) + v^4 - 2\gamma_1 v - v^2 + \frac{2\gamma_1}{v} \right] v_{xx} \\ &\quad + \left[18u_x \left(1 + \frac{\gamma_1}{v^3} \right) + 48v^3 - 24\gamma_1 - 2v - \frac{\gamma_1}{v^2} \right] v_x^2 \\ &\quad + \left[(6u_{xx} - 8uv) \left(2v - \frac{\gamma_1}{v^2} \right) - 12uu_x \left(v + \frac{\gamma_1}{v^2} \right) \right. \\ &\quad \left. - 10uv^4 + 24u^3 v - 48\gamma_1 uv + 6u_{xx} v \right] v_x \\ &\quad - (6u_x^2 + 6uu_{xx}) \left(v^2 - \frac{2\gamma_1}{v} \right) - v^2 [2u_x (v^3 + 12\gamma_1) \\ &\quad + 3(u_{xxx} + 12u^2 u_x)] - 4u_x (v^3 + \gamma_1), \\ v_t &= \frac{1}{4v^2} (v^2 v_{xxx} + 6v v_x v_{xx} + 2v_x^3) - \frac{1}{2v^2} \left(v^2 + \frac{\gamma_1}{v} \right) (uv_x)_x \\ &\quad - \frac{1}{v^2} \left[\frac{1}{4} \left(2v - \frac{\gamma_1}{v^2} \right) v_x^2 + (v^3 + \gamma_1)^2 - u^2 (v^3 + \gamma_1) \right]_x + \frac{1}{2v} [uv_x (2v - \frac{\gamma_1}{v^2})]_x. \end{aligned} \quad (8)$$

Through a direct calculation, we can derive the following result:

Proposition 1. Let $(u(x, y, t), v(x, y, t))^T$ be a compatible solution of the soliton equation (7) and (8). Then $\omega(x, y, t) = v^3 + \gamma_1$ solves the KP equation:

$$\omega_t = \frac{1}{16}(\omega_{xx} - 24\omega^2)_x + \frac{3}{4}\partial_x^{-1}\omega_{yy}. \quad (9)$$

3.The Associated Hamiltonian Systems

Consider N copies of (1) with N distinct eigenvalues $\alpha_j, 1 \leq j \leq N$, and associated eigenfunctions $\phi = (p_j, q_j)^T$, they can be written as

$$\partial_x \begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} -\alpha_j + u & 2v \\ 2(v^2 + \frac{\gamma_1}{v}) & \alpha_j - u \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}, 1 \leq j \leq N. \quad (10)$$

A direct calculation shows that $KS_j = \alpha_j JS_j$, where $S_j = (q_j^2, -p_j^2, p_j q_j)^T$. Consider the Bargmann constraint [6]

$$\gamma_1 g_{-1} + \frac{1}{2} g_1 = \sum_{j=1}^N S_j, \quad (11)$$

$$i.e. \quad \begin{cases} u = \frac{1}{3} \langle p, q \rangle^{\frac{-2}{3}} \langle q, q \rangle \\ \quad + \frac{1}{3} \langle p, p \rangle (-2 \langle p, q \rangle^{\frac{-1}{3}} + \gamma_1 \langle p, q \rangle^{\frac{-4}{3}}), \\ v = \langle p, q \rangle^{\frac{1}{3}}, \end{cases} \quad (12)$$

where $p = (p_1, \dots, p_N)^T, q = (q_1, \dots, q_N)^T$, and $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^N . Substituting (12) into (10), the spectral problem (1) can be nonlinearized to

$$\begin{aligned}
p_x &= -Ap + \left[\frac{1}{3} \langle p, q \rangle^{-\frac{2}{3}} \langle q, q \rangle + \frac{1}{3} \langle p, p \rangle (-2 \langle p, q \rangle^{-\frac{1}{3}} \right. \\
&\quad \left. + \gamma_1 \langle p, q \rangle^{-\frac{4}{3}}) \right] p + 2 \langle p, q \rangle^{\frac{1}{3}} q \\
&= -\frac{\partial H_1}{\partial q},
\end{aligned} \tag{13}$$

$$\begin{aligned}
q_x &= 2(\langle p, q \rangle^{\frac{2}{3}} + \gamma_1 \langle p, q \rangle^{-\frac{1}{3}}) p + Aq - \left[\frac{1}{3} \langle p, q \rangle^{-\frac{2}{3}} \langle q, q \rangle \right. \\
&\quad \left. + \frac{1}{3} \langle p, p \rangle (-2 \langle p, q \rangle^{-\frac{1}{3}} + \gamma_1 \langle p, q \rangle^{-\frac{4}{3}}) \right] q \\
&= \frac{\partial H_1}{\partial p},
\end{aligned}$$

where $A = \text{diag}(\alpha_1, \dots, \alpha_N)$, and

$$H_1 = \langle Ap, q \rangle + (\langle p, q \rangle^{\frac{2}{3}} + \gamma_1 \langle p, q \rangle^{-\frac{1}{3}}) \langle p, p \rangle - \langle p, q \rangle^{\frac{1}{3}} \langle q, q \rangle.$$

Constructing the solution of Lenard eigenvalue problem as

$$G_\lambda = \lambda g_{-1} + g_0 + \sum_{j=1}^N \frac{2S_j}{\lambda - \alpha_j}, \tag{14}$$

then we can obtain $(K - \lambda J)G_\lambda = 0$. Accordingly, the Lax equation (4) along the x-flow has a solution:

$$\begin{aligned}
V_\lambda &= \sigma_\lambda(G_\lambda) \\
&= \begin{pmatrix} -\lambda + 2Q_\lambda(p, q) & 2v - 2Q_\lambda(p, p) \\ 2(v^2 + \frac{\gamma_1}{v}) + 2Q_\lambda(q, q) & \lambda - 2Q_\lambda(p, q) \end{pmatrix},
\end{aligned} \tag{15}$$

where $Q_\lambda(\xi, \eta) = \sum_{j=1}^N \frac{\xi \eta}{\lambda - \alpha_j} = \sum_{n=0}^{\infty} \frac{\langle A^n \xi, \eta \rangle}{\lambda^{n+1}}$. Moreover, (4) implies that $F_\lambda = \frac{1}{4} \det V_\lambda$ is

invariant along the x-flow. Therefore, the generating function of integrals of (13) is

$$F_\lambda = \frac{1}{4} \det V_\lambda = -\left(\frac{\lambda^2}{4} + \gamma_1\right) + \sum_{n=0}^{\infty} \lambda^{-n-1} F_n, \tag{16}$$

with

$$\begin{aligned}
F_0 &= \langle Ap, q \rangle + (\langle p, q \rangle^{\frac{2}{3}} + \gamma_1 \langle p, q \rangle^{-\frac{1}{3}}) \langle p, p \rangle - \langle p, q \rangle^{\frac{1}{3}} \langle q, q \rangle, \\
F_n &= \langle A^{n+1} p, q \rangle + \sum_{i+j=n-1} \langle A^i p, p \rangle \langle A^j q, q \rangle \\
&\quad - \sum_{i+j=n-1} \langle A^i p, q \rangle \langle A^j p, q \rangle + \langle p, q \rangle^{\frac{2}{3}} \langle A^n p, p \rangle \\
&\quad + \gamma_1 \langle p, q \rangle^{-\frac{1}{3}} \langle A^n p, p \rangle - \langle p, q \rangle^{\frac{1}{3}} \langle A^n q, q \rangle.
\end{aligned} \tag{17}$$

Regarding F_λ as a Hamiltonian in the symplectic space $(R^{2N}, dp \wedge dq)$, we remind that the Poisson bracket of two smooth functions F and G is defined as [20]

$$\{F, G\} = \sum_{j=1}^N \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

Denote the conjugate variable of F_λ by τ_λ , then the canonical equation is

$$\frac{d}{d\tau_\lambda} \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_\lambda}{\partial q_k} \\ \frac{\partial F_\lambda}{\partial p_k} \end{pmatrix} = W(\lambda, \alpha_k) \begin{pmatrix} p_k \\ q_k \end{pmatrix}, \tag{18}$$

where $1 \leq k \leq N$, $W(\lambda, \alpha_k) = \frac{V_\lambda}{\lambda - \alpha_k} + V_0(\lambda)$,

$$V_0(\lambda) = \begin{pmatrix} V_0^{11} & 0 \\ 0 & V_0^{22} \end{pmatrix}$$

with

$$\begin{aligned}
V_0^{11} &= \frac{1}{3v^2} \sum_{j=1}^N \frac{q^2}{\lambda - \alpha_j} - \frac{1}{3v} \left(2 - \frac{\gamma_1}{v^3} \right) \sum_{j=1}^N \frac{p^2}{\lambda - \alpha_j} + 1, \\
V_0^{22} &= -\frac{1}{3v^2} \sum_{j=1}^N \frac{q^2}{\lambda - \alpha_j} + \frac{1}{3v} \left(2 - \frac{\gamma_1}{v^3} \right) \sum_{j=1}^N \frac{p^2}{\lambda - \alpha_j} - 1.
\end{aligned}$$

Proposition 2. The Lax matrix V_μ along the τ_λ -flow satisfies the Lax equation:

$$\frac{d}{d\tau_\lambda} V_\mu = [W(\lambda, \mu), V_\mu], \quad \forall \lambda, \mu \in \mathbb{C}. \tag{19}$$

Besides,

$$(F_\mu, F_\lambda) = 0, \quad \forall \lambda, \mu \in \mathbb{C}, \tag{20}$$

$$(F_m, F_n) = 0, \quad \forall m, n = 0, 1, 2, \dots, n \tag{21}$$

4. Relation between Conserved Integrals and Soliton Equations

We introduce a set of polynomial integrals $\{H_n\}$ for (13) as follows:

$$\begin{aligned} H_0 &= -\gamma_1, \quad H_1 = F_0, \\ H_2 &= F_1 + \gamma_1^2, \quad H_3 = -\gamma_1 F_0 + F_2, \end{aligned} \quad (22)$$

$$H_{n+1} = \sum_{\substack{s+t=n-1 \\ s,t \geq 0}} H_s H_t + F_n, \quad n = 0, 1, 2, \dots,$$

which are generated by

$$-4F_\lambda = \lambda^2(1 - 2H_\lambda)^2, \quad (23)$$

where

$$H_\lambda = \sum_{n=0}^{\infty} H_n \lambda^{-n-2}. \quad (24)$$

Proposition 3. The conserved integrals H_λ , $\{H_n\}$ of the Bargmann system (13) are involution in pairs:

$$\begin{aligned} (H_\mu, H_\lambda) &= 0, \quad \forall \lambda, \mu \in \mathbb{C}, \\ (H_m, H_n) &= 0, \quad \forall m, n = 0, 1, 2, \dots \end{aligned}$$

Exerting $J^{-1}K$ on (11) k times yields

$$\begin{aligned} 2 \sum_{j=1}^N \alpha_j^k S_j &= g_{k+1} + c_2 g_{k-1} + c_3 g_{k-2} + \dots + c_{k+2} g_{-1} \\ &= \sum_{i=0}^{k+2} c_i g_{k+1-i}, \end{aligned} \quad (25)$$

where c_j is constant of integration, and $c_0 = 1, c_1 = 0, c_2 = 2\gamma_1$. According to (25), (14) can be written as

$$G_\lambda = \lambda c_\lambda g_\lambda, \quad c_\lambda = 1 + \sum_{k=0}^{\infty} c_{k+2} \lambda^{-k-2}. \quad (26)$$

Using (15) and (16), we have

$$V_\lambda = \lambda c_\lambda \sigma_\lambda(g_\lambda), \quad 4F_\lambda = -\lambda^2 c_\lambda^2. \quad (27)$$

Then by (23), we obtain that

$$c_\lambda = 1 - 2H_\lambda. \quad (28)$$

Denote the variables of the H_λ -flow and H_n -flow by t_λ and t_n , respectively. Then we have

$$\frac{d}{dt_\lambda} = \frac{1}{\lambda^2 c_\lambda} \frac{d}{d\tau_\lambda}$$

Noticing that $\frac{dp}{d\tau_\lambda} = -\frac{\partial F_\lambda}{\partial q}$, $\frac{dq}{d\tau_\lambda} = \frac{\partial F_\lambda}{\partial p}$, under the Bargmann constraint (12), we get

$$\frac{d}{d\tau_\lambda}(u, v)^T = QPG_\lambda, \quad (29)$$

$$\frac{d}{dt_\lambda}(u, v)^T = \frac{1}{\lambda} QPg_\lambda = \sum_{n=0}^{\infty} X_{n-2} \lambda^{-n-1}. \quad (30)$$

According to (30) and (24), we can obtain the following theorem:

Theorem 1. Let $(p(x, t_n), q(x, t_n))^T$ be a compatible solution of the H_1 -, H_n -flow. Then $u(x, t_n)$ and $v(x, t_n)$ determined by (12) solve the equation

$$\frac{d}{dt_n}(u, v)^T = X_n, \quad n \geq 1.$$

Proposition 4. Let $(p(x, y, t), q(x, y, t))^T$ be a compatible solution of the H_1 -, H_2 -, H_3 -flow, that is,

$$(p, q)_x^T = I\nabla H_1, \quad (p, q)_y^T = I\nabla H_2, \quad (p, q)_t^T = I\nabla H_3,$$

where $I\nabla H_k = (-\frac{\partial H_k}{\partial q}, \frac{\partial H_k}{\partial p})^T$. Then $\omega(x, y, t) = \langle p, q \rangle + \gamma_1$ is a solution of the KP equation

(9) in which $u(x, y, t)$, $v(x, y, t)$ satisfy (12).

5. Elliptic Coordinates and the Integrability

In order to prove the functional independence of integrals, we introduce the elliptic coordinates first. According to (16), we obtain

$$F_\lambda = -\frac{1}{4}[(V_\lambda^{11})^2 + (V_\lambda^{12})(V_\lambda^{21})] = -\frac{1}{4} \frac{b(\lambda)}{a(\lambda)} = -\frac{1}{4} \frac{R(\lambda)}{a(\lambda)^2}, \quad (31)$$

$$V_\lambda^{12} = 2v - 2Q_\lambda(p, p) = 2v \frac{m(\lambda)}{a(\lambda)}, \quad (32)$$

$$V_\lambda^{21} = 2(v^2 + \frac{\gamma_1}{v}) + 2Q_\lambda(q, q) = 2(v^2 + \frac{\gamma_1}{v}) \frac{n(\lambda)}{a(\lambda)},$$

with

$$\begin{aligned}
 a(\lambda) &= \prod_{j=1}^N (\lambda - \alpha_j), b(\lambda) = \prod_{j=1}^{N+2} (\lambda - \beta_j), \\
 R(\lambda) &= a(\lambda)b(\lambda), m(\lambda) = \prod_{j=1}^N (\lambda - \mu_j), \\
 n(\lambda) &= \prod_{j=1}^N (\lambda - \nu_j).
 \end{aligned} \tag{33}$$

Here $\{\mu_j\}, \{\nu_j\}$ are called the elliptic coordinates of Hamiltonian system (13). Using (32), (33) and the Laurent expansions of $Q_\lambda(\xi, \eta)$, we can obtain

$$\begin{aligned}
 \langle p, p \rangle &= -v \sum_{j=1}^N (\alpha_j - \mu_j), \\
 \langle q, q \rangle &= (v^2 + \frac{\gamma_1}{v}) \sum_{j=1}^N (\alpha_j - \nu_j).
 \end{aligned} \tag{34}$$

With the help of (12), (13) and (34), we have

$$\frac{3}{2} \frac{v_x}{v + \frac{\gamma_1}{v^2}} = \sum_{j=1}^N (\mu_j - \nu_j), \tag{35}$$

$$\begin{aligned}
 u &= \frac{1}{3} \sum_{j=1}^N (\alpha_j - \nu_j) + \frac{\gamma_1}{3v^3} \sum_{j=1}^N (\mu_j - \nu_j) \\
 &\quad + \frac{2}{3} \sum_{j=1}^N (\alpha_j - \mu_j).
 \end{aligned} \tag{36}$$

To get the evolution of the elliptic coordinates along the τ_λ -flow, we use the components of the Lax equation (19):

$$\begin{aligned}
 \frac{dV_\mu^{12}}{d\tau_\lambda} &= 2(-\mu + 2Q_\lambda(p, q) + \Delta)V_\mu^{12} \\
 &\quad - 4(v - Q_\lambda(p, p))V_\mu^{11}, \\
 \frac{dV_\mu^{21}}{d\tau_\lambda} &= 4[(v^2 + \frac{\gamma_1}{v}) + Q_\lambda(q, q)]V_\mu^{11} \\
 &\quad - 2(-\mu + 2Q_\lambda(p, q) + \Delta)V_\mu^{21}(\mu),
 \end{aligned} \tag{37}$$

where $\Delta = \frac{1}{3v^2} \sum_{j=1}^N \frac{q^2}{\lambda - \alpha_j} - \frac{1}{3v} (2 - \frac{\gamma_1}{v^3}) \sum_{j=1}^N \frac{p^2}{\lambda - \alpha_j} + 1$. Let $\mu = \mu_k, \nu_k$, after some calculations we obtain from (32) and (37):

$$\begin{cases} \frac{1}{2\sqrt{R(\mu_k)}} \frac{d\mu_k}{d\tau_\lambda} = \frac{m(\lambda)}{a(\lambda)} \frac{1}{(\lambda - \mu_k)m'(\mu_k)}, \\ \frac{1}{2\sqrt{R(\nu_k)}} \frac{d\nu_k}{d\tau_\lambda} = -\frac{n(\lambda)}{a(\lambda)} \frac{1}{(\lambda - \nu_k)n'(\nu_k)}. \end{cases} \quad (38)$$

By applying the interpolation formula to (38), we obtain

$$\begin{aligned} \sum_{k=1}^N \frac{\mu_k^{N-j}}{2\sqrt{R(\mu_k)}} \frac{d\mu_k}{d\tau_\lambda} &= \frac{\lambda^{N-j}}{a(\lambda)}, \\ \sum_{k=1}^N \frac{\nu_k^{N-j}}{2\sqrt{R(\nu_k)}} \frac{d\nu_k}{d\tau_\lambda} &= -\frac{\lambda^{N-j}}{a(\lambda)}, \quad 1 \leq j \leq N. \end{aligned} \quad (39)$$

Consider the hyperelliptic curve Γ defined by the affine equation $\xi^2 = R(\lambda)$, with genus N .

Denote $P(\lambda) = (\lambda, \xi = \sqrt{R(\lambda)})$. For fixed $P_0 \in \Gamma$ we introduce the quasi-Abel-Jacobi coordinates:

$$\begin{aligned} \widetilde{\phi}_j &= \sum_{k=1}^N \int_{P_0}^{P(\mu_k)} \frac{\mu^{N-j}}{2\sqrt{R(\mu_k)}} d\mu, \\ \widetilde{\psi}_j &= \sum_{k=1}^N \int_{P_0}^{P(\nu_k)} \frac{\nu^{N-j}}{2\sqrt{R(\nu_k)}} d\nu, \quad 1 \leq j \leq N. \end{aligned} \quad (40)$$

Then (39) can be put in the form:

$$\frac{d\widetilde{\phi}_j}{d\tau_\lambda} = \frac{\lambda^{N-j}}{a(\lambda)}, \quad \frac{d\widetilde{\psi}_j}{d\tau_\lambda} = -\frac{\lambda^{N-j}}{a(\lambda)}, \quad 1 \leq j \leq N. \quad (41)$$

Through a standard treatment as in [7-9], we can obtain the following result:

Proposition 5. F_0, F_1, \dots, F_{N-1} given by (17) are functionally independent.

Proposition 6. H_0, H_1, \dots, H_{N-1} given by (22) are functionally independent.

Theorem 2. The finite-dimensional Hamiltonian system (13) is completely integrable in the Liouville sense.

6. The Explicit Solutions

In this section, the Abel-Jacobi coordinates will be introduced to straighten out the associated flows, from which explicit solutions of soliton equations are given. First, we take the canonical basis of cycles on $\Gamma: a_1, a_2, \dots, a_N; b_1, b_2, \dots, b_N$, which are independent and satisfy: $a_i \circ a_j = 0$, $b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$, $i, j = 1, 2, \dots, N$. It's known that

$$\widetilde{\omega}_l = \frac{\lambda^{N-l} d\lambda}{2\sqrt{R(\lambda)}}, \quad l = 1, 2, \dots, N \quad (42)$$

are N linearly independent holomorphic differentials on Γ . Let $C = (C_{il})_{N \times N}$ be the inverse of the

periodic matrix $(A_{ij})_{N \times N}$:

$$C = (A_{ij})_{N \times N}^{-1}, \quad A_{ij} = \int_{a_j} \widetilde{\omega}_i. \quad (43)$$

Then for the normalized holomorphic differential

$$\omega_i = \sum_{l=1}^N C_{il} \widetilde{\omega}_l, \quad \omega = (\omega_1, \dots, \omega_N)^T = C \widetilde{\omega}$$

we have

$$\int_{a_j} \omega_i = \delta_{ij}, \quad \int_{b_j} \omega_i = B_{ij}, \quad 1 \leq i, j \leq N, \quad (44)$$

where the matrix $B = (B_{ij})$ is symmetric with positively definite imaginary part and is used to define the Riemannian theta function of Γ [21,22]:

$$\Theta(\zeta) = \sum_{Z \in \mathbb{Z}^N} \exp(\pi i \langle BZ, Z \rangle + 2\pi i \langle \zeta, Z \rangle), \quad \zeta \in \mathbb{C}^N. \quad (45)$$

For fixed $P_0 \in \Gamma$ the Abel map $\mathcal{A}(P)$ and the Abel-Jacobi coordinates are defined as

$$\begin{aligned} \mathcal{A}(P) &= \int_{P_0}^P \omega, \quad \mathcal{A}(\sum n_k P_k) = \sum n_k \mathcal{A}(P_k). \\ \phi &= \mathcal{A}(\sum_{k=1}^N P(\mu_k)) = \sum_{k=1}^N \int_{P_0}^{P(\mu_k)} \omega, \\ \psi &= \mathcal{A}(\sum_{k=1}^N P(v_k)) = \sum_{k=1}^N \int_{P_0}^{P(v_k)} \omega. \end{aligned} \quad (46)$$

Let $S_k = \lambda_1^k + \lambda_2^k + \dots + \lambda_{2N+2}^k$ and $R_*(\lambda^{-1}) = \lambda^{-2N-2} R(\lambda) = \prod_{j=1}^{2N+2} (1 - \lambda_j \lambda^{-1})$, then the

coefficients in $R_*(\lambda^{-1})^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k}$ are determined recursively by

$$\Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{2} S_1, \quad \Lambda_k = \frac{1}{2k} (S_k + \sum_{\substack{i+j=k \\ i, j \geq 1}} S_i \Lambda_j), \quad k \geq 2.$$

With the help of (31) and (27), we have $\sqrt{R(\lambda)} = \lambda c_\lambda a(\lambda)$. Denote the k th column vector of matrix C by C_k , we have

Theorem 3. *The associated flow in Abel-Jacobi coordinate is straightened as*

$$\frac{d\phi}{dt_\lambda} = \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-2}, \quad \frac{d\psi}{dt_\lambda} = -\sum_{k=0}^{\infty} \Omega_k \lambda^{-k-2}, \quad (47)$$

$$\frac{d\phi}{dt_k} = \Omega_k, \quad \frac{d\psi}{dt_k} = -\Omega_k, \quad k \geq 0, \quad (48)$$

with the constants

$$\begin{aligned}\Omega_0 &= 0, \quad \Omega_1 = C_1, \\ \Omega_k &= (\Lambda_{k-1}C_1 + \dots + \Lambda_1C_{k-1} + C_k), \quad k \leq N-1, \\ \Omega_k &= (\Lambda_{k-1}C_1 + \dots + \Lambda_{k-N}C_N), \quad k \geq N.\end{aligned}$$

Proof:

$$\begin{aligned}\frac{d\phi}{d\tau_\lambda} &= \frac{\lambda^N}{a(\lambda)} (C_1\lambda^{-1} + C_2\lambda^{-2} + \dots + C_N\lambda^{-N}), \\ \frac{d\phi}{dt_\lambda} &= \frac{1}{\lambda^2 c_\lambda} \frac{d\phi}{d\tau_\lambda} \\ &= \frac{1}{\lambda^2 \sqrt{R_*(\lambda^{-1})}} (C_1\lambda^{-1} + C_2\lambda^{-2} + \dots + C_N\lambda^{-N}) \\ &= \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-2},\end{aligned}$$

which is exactly the first expression of (47). Similarly, we can prove the second expression of (47).

Comparing the coefficient of λ^{-k-2} in both sides of (47) yields (48).

The straighten equations (48) are easily integrated by quadratures:

$$\phi = \phi_0 + \sum_{k \geq 1} \Omega_k t_k, \quad \psi = \psi_0 - \sum_{k \geq 1} \Omega_k t_k.$$

And the evolution picture of the corresponding flows becomes very simple through the Abel-Jacobi coordinates:

$$\begin{aligned}H_k - \text{flow} : \quad & \phi = \phi_0 + \Omega_k t_k, \\ & \psi = \psi_0 - \Omega_k t_k,\end{aligned}\tag{49}$$

$$\begin{aligned}X_k - \text{flow} : \quad & \phi = \phi_0 + \Omega_1 x + \Omega_k t_k, \\ & \psi = \psi_0 - \Omega_1 x - \Omega_k t_k.\end{aligned}\tag{50}$$

(50) gives the explicit solution of the soliton equation (6) and the KP equation (9), in the Abel-Jacobi coordinates (ϕ, ψ) .

Next, we shall give the solution in the original coordinates. According to the Riemann theorem, there exist Riemann constants $M_1, M_2 \in \mathbb{C}^N$, such that $\Theta(\mathcal{A}(P(\lambda)) - \phi - M_1)$ has exactly N zeros at $\lambda = \mu_1, \dots, \mu_N$; and $\Theta(\mathcal{A}(P(\lambda)) - \psi - M_2)$ has exactly N zeros at $\lambda = \nu_1, \dots, \nu_N$ as well. For the same λ , there are two points $(\lambda, \sqrt{R(\lambda)})$ and $(\lambda, -\sqrt{R(\lambda)})$ on the upper and lower sheets of Γ , respectively. Thus in the local coordinates $z = \lambda^{-1}$ near the infinities $\infty_s, s = 1, 2$, we have

$$\mathcal{A}(P(z^{-1})) = -\eta_s + \frac{(-1)^{s-1}}{2} \sum_{k=1}^{\infty} \frac{1}{k} \Omega_k z^k,$$

with $\eta_s = \int_{\infty_s}^{P_0} \omega$. Through a standard treatment [9,10,23], we obtain

$$\sum_{j=1}^N \mu_j = I_1(\Gamma) + \frac{1}{2} \partial \ln \frac{\Theta(\phi + M_1 + \eta_1)}{\Theta(\phi + M_1 + \eta_2)}, \quad (51)$$

$$\sum_{j=1}^N \nu_j = I_1(\Gamma) - \frac{1}{2} \partial \ln \frac{\Theta(-\psi - M_2 - \eta_1)}{\Theta(-\psi - M_2 - \eta_2)}, \quad (52)$$

with $I_1(\Gamma) = \sum_{j=1}^N \int_{a_j} \lambda \omega_j$. Noticing the H_1 -, H_n -flows and the evolution of X_n -flow (50),

with the aid of (35) and (36), the soliton equation (6) have quasi-periodic solutions:

$$\begin{aligned} v^3(x, t_n) &= (v^3(0, t_n) + \gamma_1) v_{11} v_{12}, \\ u(x, t_n) &= \frac{1}{3} \partial \ln \frac{\Theta(\Omega_1 x + \Omega_n t_n + \beta_1)}{\Theta(\Omega_1 x + \Omega_n t_n + \beta_2)} \\ &\quad - \frac{2}{3} \partial \ln \frac{\Theta(\Omega_1 x + \Omega_n t_n + \alpha_1)}{\Theta(\Omega_1 x + \Omega_n t_n + \alpha_2)} \\ &\quad + \frac{\gamma_1}{6v^3(x, t_n)} \partial \ln v_{11} v_{12} + \kappa_0, \end{aligned}$$

where

$$\begin{aligned} \kappa_0 &= \sum_{j=1}^N \alpha_j - I_1(\Gamma), \\ v_{11} &= \frac{\Theta(\Omega_1 x + \Omega_n t_n + \alpha_1) \Theta(\Omega_1 x + \Omega_n t_n + \beta_1)}{\Theta(\Omega_1 x + \Omega_n t_n + \alpha_2) \Theta(\Omega_1 x + \Omega_n t_n + \beta_2)}, \\ v_{12} &= \frac{\Theta(\Omega_n t_n + \beta_2)}{\Theta(\Omega_n t_n + \beta_1)} - \gamma_1, \end{aligned}$$

with constants $\alpha_l = \phi_0 + M_1 + \eta_l$, $\beta_l = -\psi_0 - M_2 - \eta_l$, $l = 1, 2$. In a similar way, we obtain quasi-periodic solution of the KP equation (9)

$$\omega(x, y, t) = \omega(0, y, t) \frac{\omega_{11}}{\omega_{22}} \Theta(\Omega_2 y + \Omega_3 t + \alpha_2) \Theta(\Omega_2 y + \Omega_3 t + \beta_2),$$

where

$$\begin{aligned} \omega_{11} &= \Theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + \alpha_1) \Theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + \beta_1), \\ \omega_{22} &= \Theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + \alpha_2) \Theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + \beta_2). \end{aligned}$$

7. Conclusions

In this paper the Kadomtsev-Petviashvili (KP) equation is separated into systems of compatible ordinary differential equations with the help of two (1+1)-dimensional soliton equations. Quasi-periodic solution of the KP equation is finally obtained in terms of Riemann theta functions. During that course, the generating function approach is used to prove the involutivity and the functional independence of the conserved integrals, and the Abel-Jacobi coordinates are introduced to linearize the associated flows.

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